

A FULLY ANISOTROPIC MECHANISM FOR FORMATION OF TRAPPED SURFACES IN VACUUM

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ABSTRACT. We present a new, fully anisotropic, criterion for formation of trapped surfaces in vacuum. More precisely we provide conditions on null data, concentrated in a neighborhood of a short null geodesic segment (possibly flat everywhere else) whose future development contains a trapped surface. This extends considerably the previous result of Christodoulou [2] which required instead a uniform condition along all null geodesic generators. To obtain our result we combine Christodoulou's mechanism for the formation of a trapped surface with a new deformation process which takes place along incoming null hypersurfaces.

1. INTRODUCTION

According to the celebrated incompleteness result of Penrose, the future Cauchy development of a non-compact, complete, initial data set of the Einstein vacuum¹ equations which contains a *trapped* surface, must be incomplete. Thus, in a sense, the fundamental issue of formation of spacetime singularities in gravitational collapse is reduced to the somewhat more tangible problem of formation of trapped surfaces. This, on the other hand, is still a highly non-trivial problem. Indeed, the expansions of both null geodesic congruences generated by a compact, trapped surface S is required, by definition, to be negative at every point on S . To show that such surfaces can form in evolution, starting with regular initial data sets which contain no trapped surfaces, requires a deep understanding of the dynamics of the Einstein equations. It is for this reason that the problem has remained open for more than forty year, in the wake of Penrose's result, until the recent breakthrough of Christodoulou. In [2] he was able to identify an open set of regular² initial conditions, on a finite outgoing null hypersurface, with trivial data on an incoming null hypersurface, whose future development must contain a trapped surface. The main condition in Christodoulou's result is that the data verify a uniform lower bound condition, with respect to all short, null geodesic generators of the outgoing initial null hypersurface. The goal of this paper is to significantly relax this uniform condition by showing that a trapped surface forms even if the null outgoing data is only concentrated

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¹The result of Penrose applies in fact to the more general Einstein-matter equations satisfying the null energy condition, but we restrict our considerations here to the vacuum case.

²Smooth and free of trapped surfaces.

in a neighborhood of a short null geodesic segment (possibly flat everywhere else).

We recall that Christodoulou's proof in [2] rests on two main ingredients:

- (1) A semi-global existence result for the characteristic initial value problem with large initial data³ measured relative to a small parameter $\delta > 0$. The precise dependence on δ , which Christodoulou calls the short pulse method, was subsequently relaxed in [5], [6], see also [8]. In all these results the data on the incoming null hypersurface is assumed to be flat. This restriction has been recently removed in [7].
The semi-global result allows one to construct the future development of the initial data, together with a double null foliation⁴ (u, \underline{u}) , $0 \leq u \leq u_*$, $0 \leq \underline{u} \leq \underline{u}_*$, and full control on all the geometric quantities associated to it.
- (2) An amplification mechanism for the integrals $\int_\gamma |\hat{\chi}|^2$ along outgoing null geodesic segments γ , (with $\hat{\chi}$ denoting the outgoing null shear). This mechanism, which requires the estimates obtained in the constructive step (1), combines with a uniform lower bound assumption of these integrals on the initial null hypersurface, and leads to the formation of a trapped surface. It is important to note that the trapped surface thus obtained is adapted to the double null foliation (u, \underline{u}) , i.e. it is of the form $S = \{u = u_1, \underline{u} = \underline{u}_1\}$, for some $0 < u_1 \leq u_*$, $0 < \underline{u}_1 \leq \underline{u}_*$.

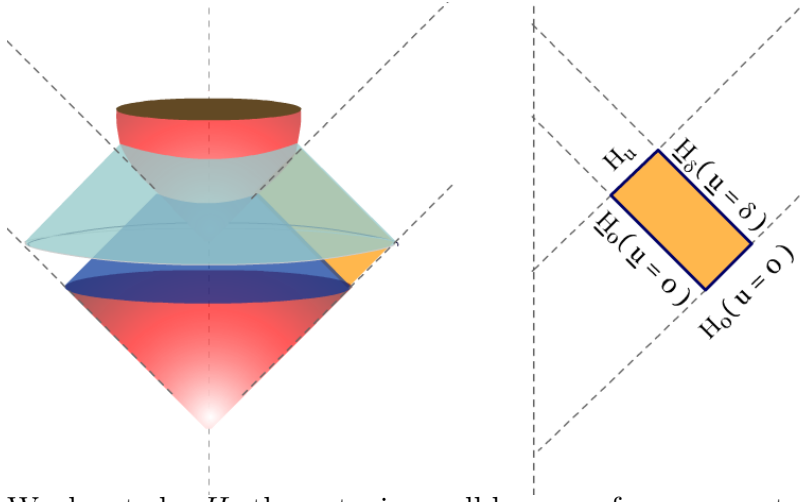
The new result we present in this paper relies heavily on the *hard* part of the above results, i.e. the construction of the spacetime in (1). We modify however part (2) by combining Christodoulou's argument with a new deformation argument along the incoming null hypersurfaces $\{\underline{u} = \text{const}\}$. This allows us to dramatically weaken his uniform condition to a merely localized condition in a neighborhood of a null geodesic of $\{u = 0\}$. We note that the trapped surface we find by our argument does not belong any longer to the double null foliation constructed in step (1).

1.1. Geometry of a double null foliation. As in [5] we consider a region $\mathcal{D} = \mathcal{D}(u_*, \underline{u}_*)$ of a vacuum spacetime (M, g) spanned by a double null foliation generated by the optical functions (u, \underline{u}) increasing towards the future⁵, $0 \leq u \leq u_*$ and $0 \leq \underline{u} \leq \underline{u}_*$ (see Figure 1).

³On the outgoing null hypersurface. The incoming data is flat.

⁴Such that the initial configuration is given by the incoming $\{\underline{u} = 0\}$ and outgoing $\{u = 0\}$ initial null hypersurfaces.

⁵These can be compared to the optical functions $u = \frac{t-r+r_0}{2}$, $\underline{u} = \frac{t+r-r_0}{2}$ in Minkowski spacetime.



The colored region on the right represents the domain $\mathcal{D}(u_*, \underline{u}_*)$, $\underline{u}_* = \delta$. The same picture is represented more realistically on the left.

We denote by H_u the outgoing null hypersurfaces generated by the level surfaces of u and by $\underline{H}_{\underline{u}}$ the incoming null hypersurfaces generated level hypersurfaces of \underline{u} . We write $S_{u,\underline{u}} = H_u \cap \underline{H}_{\underline{u}}$ and denote by $H_u^{(\underline{u}_1, \underline{u}_2)}$ and $\underline{H}_{\underline{u}}^{(u_1, u_2)}$ the regions of these null hypersurfaces defined by $\underline{u}_1 \leq \underline{u} \leq \underline{u}_2$ and respectively $u_1 \leq u \leq u_2$. Let $L = -g^{\alpha\beta} \partial_\alpha u \partial_\beta$, $\underline{L} = -g^{\alpha\beta} \partial_\alpha \underline{u} \partial_\beta$, be the geodesic vectorfields associated to the two foliations and define,

$$g(L, \underline{L}) := -2\Omega^{-2} = g^{\alpha\beta} \partial_\alpha u \partial_\beta \underline{u} \quad (1)$$

As is well known, the space-time slab $\mathcal{D}(u_*, \underline{u}_*)$ is completely determined (for small values of u_*, \underline{u}_*) by data along the null hypersurfaces H_0, \underline{H}_0 corresponding to $\underline{u} = 0$ and $u = 0$ respectively. We can construct our double null foliation such that $\Omega = 1$ along H_0 and \underline{H}_0 , i.e.,

$$\Omega(0, \underline{u}) = 1, \quad 0 \leq \underline{u} \leq \underline{u}_*, \quad (2)$$

$$\Omega(u, 0) = 1, \quad 0 \leq u \leq u_*. \quad (3)$$

We denote by $r = r(u, \underline{u})$ the radius of the 2-surfaces $S = S_{u,\underline{u}}$, i.e. $|S_{u,\underline{u}}| = 4\pi r(u, \underline{u})^2$. We denote by r_0 the value of r for $S_{0,0}$, i.e. $r_0 = r(0, 0)$.

Throughout this paper we work with the normalized null pair (e_3, e_4) defined by

$$e_3 = \Omega \underline{L}, \quad e_4 = \Omega L$$

which satisfy

$$g(e_3, e_4) = -2.$$

Given a 2-surfaces $S_{u,\underline{u}}$ and $(e_a)_{a=1,2}$ an arbitrary frame tangent to it we define the Ricci coefficients,

$$\Gamma_{(\lambda)(\mu)(\nu)} = g(e_{(\lambda)}, D_{e_{(\mu)}} e_{(\nu)}), \quad \lambda, \mu, \nu = 1, 2, 3, 4 \quad (4)$$

These coefficients are completely determined by the following components,

$$\begin{aligned}\chi_{ab} &= g(D_a e_4, e_b), & \underline{\chi}_{ab} &= g(D_a e_3, e_b), \\ \eta_a &= -\frac{1}{2}g(D_3 e_a, e_4), & \underline{\eta}_a &= -\frac{1}{2}g(D_4 e_a, e_3) \\ \omega &= -\frac{1}{4}g(D_4 e_3, e_4), & \underline{\omega} &= -\frac{1}{4}g(D_3 e_4, e_3), \\ \zeta_a &= \frac{1}{2}g(D_a e_4, e_3)\end{aligned}\tag{5}$$

where $D_a = D_{e_{(a)}}$. We also introduce the null curvature components,

$$\begin{aligned}\alpha_{ab} &= R(e_a, e_4, e_b, e_4), & \underline{\alpha}_{ab} &= R(e_a, e_3, e_b, e_3), \\ \beta_a &= \frac{1}{2}R(e_a, e_4, e_3, e_4), & \underline{\beta}_a &= \frac{1}{2}R(e_a, e_3, e_3, e_4), \\ \rho &= \frac{1}{4}R(Le_4, e_3, e_4, e_3), & \sigma &= \frac{1}{4}{}^*R(e_4, e_3, e_4, e_3)\end{aligned}\tag{6}$$

Here *R denotes the Hodge dual of R . We denote by ∇ the induced covariant derivative operator on $S_{u, \underline{u}}$ and by ∇_3, ∇_4 the projections to $S_{u, \underline{u}}$ of the covariant derivatives D_3, D_4 . Observe that,

$$\begin{aligned}\omega &= -\frac{1}{2}\nabla_4(\log \Omega), & \underline{\omega} &= -\frac{1}{2}\nabla_3(\log \Omega), \\ \eta_a &= \zeta_a + \nabla_a(\log \Omega), & \underline{\eta}_a &= -\zeta_a + \nabla_a(\log \Omega)\end{aligned}\tag{7}$$

We recall the integral formulas⁶ for a scalar function f in \mathcal{D} ,

$$\begin{aligned}\frac{d}{d\underline{u}} \int_{S_{u, \underline{u}}} f &= \int_{S_{u, \underline{u}}} \left(\frac{df}{d\underline{u}} + \Omega \text{tr} \chi f \right) = \int_{S_{u, \underline{u}}} \Omega (e_4(f) + \text{tr} \chi f) \\ \frac{d}{du} \int_{S_{u, \underline{u}}} f &= \int_{S_{u, \underline{u}}} \left(\frac{df}{du} + \Omega \text{tr} \underline{\chi} f \right) = \int_{S_{u, \underline{u}}} \Omega (e_3(f) + \text{tr} \underline{\chi} f)\end{aligned}\tag{8}$$

In particular,

$$\frac{dr}{d\underline{u}} = \frac{1}{8\pi} \int_{S_{u, \underline{u}}} \Omega \text{tr} \chi, \quad \frac{dr}{du} = \frac{1}{8\pi} \int_{S_{u, \underline{u}}} \Omega \text{tr} \underline{\chi}\tag{9}$$

We also recall the following commutation formulae between ∇ and ∇_4, ∇_3 in [4]:

Lemma 1. *For a scalar function f :*

$$[\nabla_4, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})D_4 f - \chi \cdot \nabla f\tag{10}$$

$$[\nabla_3, \nabla]f = \frac{1}{2}(\eta + \underline{\eta})D_3 f - \underline{\chi} \cdot \nabla f,\tag{11}$$

⁶see for example Lemma 3.1.3 in [4]

For a 1-form tangent to S :

$$\begin{aligned} [\nabla_4, \nabla_a]U_b &= -\chi_{ac}\nabla_c U_b + \epsilon_{ac} {}^*\beta_b U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)D_4 U_b \\ &\quad - \chi_{ac}\underline{\eta}_b U_c + \chi_{ab}\underline{\eta} \cdot U \\ [\nabla_3, \nabla_a]U_b &= -\underline{\chi}_{ac}\nabla_c U_b + \epsilon_{ac}^* \underline{\beta}_b U_c + \frac{1}{2}(\eta_a + \underline{\eta}_a)D_3 U_b \\ &\quad - \underline{\chi}_{ac}\eta_b U_c + \underline{\chi}_{ab}\eta \cdot U \end{aligned}$$

In particular,

$$\begin{aligned} [\nabla_4, \text{div}]U &= -\frac{1}{2}\text{tr}\chi \text{div} U - \hat{\chi} \cdot \nabla U - \beta \cdot U + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_4 U - \underline{\eta} \cdot \hat{\chi} \cdot U \\ [\nabla_3, \text{div}]U &= -\frac{1}{2}\text{tr}\underline{\chi} \text{div} U - \underline{\hat{\chi}} \cdot \nabla U + \underline{\beta} \cdot U + \frac{1}{2}(\eta + \underline{\eta}) \cdot \nabla_3 U - \eta \cdot \underline{\hat{\chi}} \cdot U \end{aligned}$$

1.2. Main theorem. For simplicity of our presentation⁷ we describe our main result in the context of the class of initial data used by Christodoulou. As in [2] we prescribe the null incoming data to be trivial, i.e. corresponding to null cones in Minkowski space. In particular $S_{0,0}$ is the standard sphere of radius r_0 . To prescribe null data on H_0 amounts to prescribe an arbitrary symmetric, traceless, smooth tensor $\hat{\chi}_0$, called initial shear.

Definition 1. Given $\delta > 0$ and $C > 0$, we say that a smooth shear $\hat{\chi}_0$, supported on $H_0^{(0,\delta)}$, verifies Christodoulou's δ -short pulse condition with constant C if,

$$\sup_{\underline{u}} \sum_{i \leq 5} \sum_{k \leq 3} \delta^{\frac{1}{2}+k} \|\nabla_4^k \nabla^i \hat{\chi}_0\|_{L^\infty(S_{0,\underline{u}})} \leq C. \quad (12)$$

It is proved in [2] that for δ sufficiently small, depending only on C , r_0 , any prescribed shear χ_0 satisfying (12) gives rise to a unique smooth spacetime $\mathcal{D}(u_*, \delta)$. To show that a trapped surface forms in $\mathcal{D}(u_*, \delta)$, Christodoulou needs in addition a uniform lower bound on the function $M_0 = M_0[\hat{\chi}_0]$ defined on $S_{0,0}$ as follows,

$$M_0(\omega) = M_0[\hat{\chi}_0](\omega) := \int_0^\delta |\hat{\chi}_0|^2(\underline{u}', \omega) d\underline{u}'. \quad (13)$$

where the integral is taken along the null geodesic generator on H_0 , transversal to $S_{0,0}$, initiating at $\omega \in S_{0,0}$. More precisely he imposes the condition,

$$\inf_{\omega \in S_{0,0}} M_0(\omega) \geq M_* > 0 \quad (14)$$

and shows that if δ is sufficiently small, depending only on C , r_0 and M_* , the surface $S_{\underline{u}_*, \delta}$ is necessarily trapped. In our main result, stated below, we replace (14) by the following weaker condition,

$$\inf_{\omega \in B_p(\epsilon)} M_0(\omega) \geq M_* > 0, \quad (15)$$

⁷Similar results can be derived using the classes of initial data discussed in [5], [6] and [7].

where $B_p(\epsilon)$ is a geodesic ball of radius ϵ around some $p \in S_{0,0}$. We are now ready to state our main result.

Theorem 1 (Main theorem). *For every r_0, C, M_*, ϵ , there exists $\delta > 0$ sufficiently small such that if the null shear $\hat{\chi}_0$ along H_0 verifies both Christodoulou's δ -short pulse condition with constant C and the non-isotropic condition (15) with constants ϵ and M_* then,*

(1) *There exists a unique smooth spacetime $\mathcal{D}(u_*, \delta)$, where u_* verifies:*

$$\frac{M_* \epsilon^2}{r_0^2} \gg (r_0 - u_*) > 0, \quad (16)$$

*such that, in particular⁸, the conditions **MA1** - **MA4** below are satisfied, with $\delta_0 = \delta^{1/2}$.*

(2) *The spacetime $\mathcal{D}(u_*, \delta)$ contains a trapped surface of area at least $\approx (r_0 - u_*)^2$.*

As discussed above, part (1) of our main theorem is a consequence of Christodoulou's existence result in [2]. More precisely for every⁹ $u_* < r_0$, fixed $C > 0$, and $\delta > 0$ sufficiently small, depending only on u_*, r_0 and C , Christodoulou's theorem produces a space-time $\mathcal{D}(u_*, \delta)$ which verifies the following conditions, with $\delta_0 = \delta^{1/2}$.

MA1. Ω is comparable with its initial value

$$\Omega = 1 + O(\delta_0).$$

MA2. The Ricci coefficients $\chi, \omega, \eta, \underline{\eta}, \nabla(\log \Omega), \underline{\chi}, \underline{\omega}$ verify

$$\begin{aligned} |\hat{\chi}, \omega| &= O(\delta^{-1/2}), \\ |\text{tr} \chi| &= O(1), \\ |\eta, \underline{\eta}, \hat{\chi}, \text{tr} \underline{\chi} + \frac{2}{r}, \underline{\omega}, \nabla(\log \Omega)| &= O(\delta_0). \end{aligned}$$

MA3. Also assume that the following hold for the derivatives of Ricci coefficients

$$\begin{aligned} |\nabla \eta| &= O(\delta_0 \delta^{-1/2}), \\ |\nabla \hat{\chi}, \nabla \text{tr} \underline{\chi}, \nabla \underline{\omega}| &= O(\delta_0). \end{aligned}$$

⁸The existence proof produces a space-time with explicit control of both Ricci coefficients and null curvature components of the double null foliation. Conditions **MA1** - **MA4** are the minimum needed to be able to implement part 2 of the theorem.

⁹ $r_0 - u_* > 0$ can be made arbitrarily small by choosing $\delta > 0$ sufficiently small. Though strictly speaking, this statement that $r_0 - u_*$ can be made small by choosing δ small is not proved in [2], it is easily implied by its method of proof.

MA4. $\text{tr}\chi$ is close to its Minkowskian value on the initial cone \underline{H}_0

$$|\text{tr}\chi - \frac{2}{r}| = O(\delta_0), \quad \text{on } \underline{H}_0.$$

Remark 1. The conditions **MA1**- **MA4** are precisely the conditions needed to implement part (2) of our main theorem. Though strictly speaking Christodoulou's theorem implies **MA1**- **MA4** with $\delta_0 = \delta^{1/2}$ we prefer to write them in this more general form with respect to a second parameter δ_0 , such that $\delta_0, \delta_0^{-1}\delta$ are sufficiently small. This formulation allows us to adapt our result to the more general initial data used in [5] and [6].

Remark 2. As described above our condition (15) provides a significant relaxation of Christodoulou's uniform condition in [2], which requires

$$\inf_{\omega \in S_{0,0}} M_0(\omega) \geq M_* > 0.$$

Note however that while in [2] the desired trapped surface can be found among the surfaces $S_{u,\underline{u}}$, consistent with the double null foliation, this is no longer the case in our theorem.

Remark 3. We would like to emphasize that in our main theorem M_* and ϵ can be chosen arbitrarily small, as long as δ is sufficiently small depending on M_* and ϵ . Note also that the condition (16) is consistent with the spirit of Penrose inequality which require that the square root of the area is bounded from above by the total initial energy $\int_{H_0} |\hat{\chi}|^2$.

We show that a trapped surface exists in $\mathcal{D}(u_*, \delta)$ by finding an embedded trapped 2-sphere in the incoming null hypersurface \underline{H}_δ . Notice that by **MA2**, $\text{tr}\chi < 0$ on \underline{H}_δ . Therefore, it suffices to find a 2-sphere such that the outgoing null expansion is also pointwise negative. We will achieve this in two steps. In Section 2 we make use of the conditions **MA1** - **MA4** to reduce the problem of existence of a trapped surface to that of finding appropriate solutions to an elliptic inequality, see (20), on $S_{0,0}$. In Section 3, we prove that under the assumption (15) of the main theorem, a desired solution to the elliptic inequality exists.

2. REDUCTION TO AN ELLIPTIC INEQUALITY ON THE INITIAL SPHERE $S_{0,0}$

In this section, we show that under the assumptions **MA1** - **MA4**, the existence of a trapped surface can be reduced to constructing a solution to an elliptic inequality (20). The main result of the section is stated in Theorem 2.

We first make a remark concerning the global parametrization of points in $\mathcal{D}(u_*, \delta)$ by r , \underline{u} and coordinates ω on $S_{0,0}$.

In view of (9) and the conditions **MA1**- **MA2** we have,

$$\frac{dr}{du} = -1 + O(r\delta_0), \quad \frac{dr}{d\underline{u}} = O(r). \quad (17)$$

Thus, for δ_0 sufficiently small, we infer that r is a strictly decreasing function in u along the incoming null hypersurfaces.

Proceeding as in [2] we associate to each coordinate patch on $S_{0,0}$, a system of transported coordinates defined by

$$\mathcal{L}_{\Omega_{e_4}} \theta^a = 0, \quad \text{on } H_0, \quad (18)$$

and

$$\mathcal{L}_{\Omega_{e_3}} \theta^a = 0, \quad \text{in } \mathcal{D}(u_*, \delta), \quad (19)$$

where \mathcal{L} is the restriction of the Lie derivative to $TS_{u,\underline{u}}$ (see [2], chapter 1). This provides an identification of each point in the spacetime $\mathcal{D}(u_*, \delta)$ with a point in the initial sphere $S_{0,0}$ by the value of the coordinate functions.

It follows that any point in $\mathcal{D}(u_*, \delta)$ can also be uniquely specified by the coordinates $(\underline{u}, r, \omega)$, where $\omega \in S_{0,0}$.

We can now state the main result of this section:

Theorem 2. *Assume that the spacetime $\mathcal{D}(u_*, \delta)$ satisfies **MA1** - **MA4**. Let M_0 be the function on the initial sphere $S_{0,0}$ defined by (13), i.e.,*

$$M_0(\omega) := \int_0^\delta |\hat{\chi}|^2(u = 0, \underline{u}', \omega) d\underline{u}'.$$

Assume R is a smooth function on $S_{0,0}$ satisfying $r_0 - u_ + C\delta_0 < R < r_0$ as well as the elliptic inequality¹⁰ on $S_{0,0}$*

$$-\Delta R + R^{-1}|\nabla R|^2 + R < 2^{-1}M_0 - C\delta_0, \quad (20)$$

with $C > 0$ a constant depending only on $\sum_{i \leq 2} \|\nabla^i R\|_{L^\infty(S_{0,0})}$.

Then, for $\delta\delta_0^{-1}$, δ_0 sufficiently small, the 2-sphere defined by $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ is a trapped surface.

We prove Theorem 2 in this section and leave for the next section the task to show that a smooth solution R to (20) exists, provided that M_0 satisfies the assumptions of our main theorem.

¹⁰Here Δ and ∇ are defined with respect to the connection on the initial sphere $S_{0,0}$.

2.1. Christodoulou's argument. In [2], it was shown that under the assumptions **MA1** - **MA4**, the expansion $\text{tr}\chi$ on each of the spheres $S_{u,\delta}$ on \underline{H}_δ can be computed up to a small error depending on δ . In the context of [2], where a uniform lower bound on M_0 is assumed, this is sufficient to conclude the existence of a trapped surface S of the form $S = S_{u,\delta}$. In the case of our weaker condition (15), his argument only shows that $\text{tr}\chi$ becomes sufficiently negative in part of the sphere $S_{u,\delta}$. To obtain a trapped surface we need to combine that fact with a new deformation argument of the foliation on \underline{H}_δ .

Christodoulou's argument for the formation of trapped surfaces in [2] rests on the equations,

$$\begin{aligned}\nabla_4 \text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 &= -|\hat{\chi}|^2 - \frac{1}{2}(\text{tr}\chi)^2 - 2\omega \text{tr}\chi \\ \nabla_3 \hat{\chi} + \frac{1}{2}\text{tr}\chi \hat{\chi} &= \nabla \hat{\otimes} \eta + 2\underline{\omega} \hat{\chi} - \frac{1}{2}\text{tr}\chi \hat{\chi} + \eta \hat{\otimes} \eta\end{aligned}$$

In view of our Ricci coefficients assumptions we can rewrite,

$$\begin{aligned}\nabla_4 \text{tr}\chi &= -|\hat{\chi}|^2 + O(\delta^{-1/2}) \\ \nabla_3 \hat{\chi} + \frac{1}{2}\text{tr}\chi \hat{\chi} &= O(\delta_0 \delta^{-1/2})\end{aligned}$$

Multiplying the second equation by $\hat{\chi}$,

$$\nabla_4 |\hat{\chi}|^2 + \text{tr}\chi |\hat{\chi}|^2 = O(\delta_0 \delta^{-1})$$

Using also our assumptions for u, \underline{u}, Ω we deduce,

$$\frac{d}{d\underline{u}} \text{tr}\chi = -|\hat{\chi}|^2 + O(\delta^{-1/2}) \quad (21)$$

$$\frac{d}{d\underline{u}} |\hat{\chi}|^2 + \text{tr}\chi |\hat{\chi}|^2 = O(\delta_0 \delta^{-1}) \quad (22)$$

Integrating (21) we obtain,

$$\text{tr}\chi(u, \underline{u}) = \frac{2}{r(u, 0)} - \int_0^{\underline{u}} |\hat{\chi}|(u, \underline{u}')^2 d\underline{u}' + O(\delta_0) \quad (23)$$

In view of our assumptions for $\text{tr}\chi$ and $\frac{dr}{du}$, (22) implies

$$\begin{aligned}\frac{d}{du}(r^2 |\hat{\chi}|^2) &= r^2 \frac{d}{du} |\hat{\chi}|^2 + 2r \frac{dr}{du} |\hat{\chi}|^2 = r^2 [-\text{tr}\chi |\hat{\chi}|^2 + O(\delta_0 \delta^{-1})] + 2r [-1 + O(r\delta_0)] |\hat{\chi}|^2 \\ &= O(\delta_0 \delta^{-1}).\end{aligned}$$

Therefore,

$$r^2 |\hat{\chi}|^2(u, \underline{u}) = r^2(0, \underline{u}) |\hat{\chi}|^2(0, \underline{u}) + O(\delta_0 \delta^{-1})$$

Let $\hat{\chi}_0$ denote the initial data for $\hat{\chi}$:

$$\hat{\chi}_0(\underline{u}) = \hat{\chi}(0, \underline{u}). \quad (24)$$

We deduce,

$$|\hat{\chi}|^2(u, \underline{u}) = \frac{r^2(0, \underline{u})}{r^2(u, \underline{u})} |\hat{\chi}_0|^2(\underline{u}) + O(\delta_0 \delta^{-1})$$

Since $r(u, \underline{u}) = r(u, 0) + O(\delta)$,

$$|\hat{\chi}|^2(u, \underline{u}) = \frac{r_0^2}{r^2(u, 0)} |\hat{\chi}_0|^2(\underline{u}) + O(\delta_0 \delta^{-1}).$$

Thus, returning to (23), and recalling that

$$M_0(\omega) = \int_0^\delta |\hat{\chi}_0|^2(\underline{u}', \omega) d\underline{u}',$$

we deduce the following:

Proposition 1. *Under the assumptions **MA1-MA4** we have, for $\delta_0, \delta_0^{-1}\delta$ sufficiently small,*

$$\text{tr}\chi(u, \underline{u} = \delta, \omega) = \frac{2}{r(u, 0)} - \frac{r_0^2}{r^2(u, 0)} M_0(\omega) + O(\delta_0) \quad (25)$$

Since $r(u, 0) = r_0 - u + O(\delta_0)$, this implies

Corollary 1. *For $\delta_0, \delta_0^{-1}\delta$ small, the necessary and sufficient condition to have $\text{tr}\chi < 0$ everywhere on the sphere $S_{u, \delta}$ is that*

$$\frac{2(r_0 - u)}{r_0^2} < M_0(\omega) - O(\delta_0) \quad (26)$$

holds uniformly for every $\omega \in S_{0,0}$.

Under the assumptions of our main theorem, we can only hope that the outgoing null expansion $\text{tr}\chi$ adapted to the foliation (u, \underline{u}) becomes negative in the part where M_0 is positive. Thus to prove our main theorem, we need to combine this argument with the new deformation mechanism which leads to the formation of a trapped surface that is no longer adapted to the double null foliation (u, \underline{u}) . Instead, as stated in Theorem 2, the trapped surface will be a topological 2-sphere embedded in the incoming null hypersurface $\{\underline{u} = \delta\}$ defined by $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$.

2.2. Main transformation formula. According to the statement of Theorem 2, $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ will correspond to a trapped surface provided that R satisfies (20). To verify that, we need to compute its null expansion, which differs from the null expansion $\text{tr}\chi$ relative to the double null foliation (u, \underline{u}) restricted to $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$. To compute the correct null expansion $\text{tr}\chi'$, we change the u foliation along $\{\underline{u} = \delta\}$. More precisely, given the foliation induced by u , we look for a new foliation $v = v(u, \omega)$ defined by the equations

$$\begin{aligned} \nabla_u v &= e^f, & v|_{S_0} &= u|_{S_0} = 0 \\ \nabla_u f &= 0, & f|_{S_0} &= f_0 \end{aligned} \quad (27)$$

with

$$\nabla_u = \Omega \nabla_3,$$

and f_0 a function on $S_0 = S_{0,\delta}$ such that the sphere $\{(u, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ corresponds to the level set of v : $\{\underline{u} = \delta, v = 1\}$.

We introduce the new null frame adapted to the v -foliation,

$$e'_3 = e_3, \quad e'_a = e_a - e^{-f} \Omega e_a(v) e_3, \quad e'_4 = e_4 - 2e^{-f} \Omega e_a(v) e_a + e^{-2f} \Omega^2 |\nabla v|^2 e_3 \quad (28)$$

Indeed we have, $e'_a(v) = e_a(v) - e^{-f} \Omega e_a(v) e_3(v) = e_a(v) - e^{-f} e_a(v) \nabla_u(v) = 0$. Also, since e_3 is orthogonal to any vector tangent to \underline{H} we easily check that

$$g(e'_a, e'_b) = g(e_a, e_b) = \delta_{ab}, \quad g(e'_4, e'_a) = g(e'_4, e'_4) = 0, \quad g(e'_3, e'_4) = -2.$$

We prove the following.

Proposition 2. *Let v, f be defined according to (27), $F = \Omega^{-1} e^{-f} \nabla v$ and $G = e^{-f} F$. The trace of the null second fundamental form χ' , relative to the new frame (28), is given by*

$$tr \chi' = tr \chi - 2e^f \operatorname{div} G - tr \chi |F|^2 - 4\hat{\chi}_{bc} F^b F^c - 2(\eta + \zeta) \cdot F \quad (29)$$

where f obeys the transport equation

$$\nabla_3 f = 0, \quad (30)$$

F verifies the transport equation

$$\nabla_3 F + \frac{1}{2} tr \chi F = \nabla f - \hat{\chi} \cdot F + 2\omega F \quad (31)$$

and $\operatorname{div} G$ verifies,

$$\nabla_3(\operatorname{div} G) + tr \chi \operatorname{div} G = \operatorname{div} (e^{-f} \nabla f) + Err_1 \quad (32)$$

where,

$$\begin{aligned} Err_1 &= e^{-f} \nabla(\log \Omega) \cdot \nabla f - 2\hat{\chi} \cdot \nabla G \\ &- \nabla tr \chi \cdot G + (tr \chi \zeta - 2\hat{\chi} \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega) + 2\nabla \omega + 2\omega \nabla \log \Omega) \cdot G \end{aligned}$$

Also,

$$\nabla_3[e^f \operatorname{div} (e^{-f} \nabla f)] + tr \chi[e^f \operatorname{div} (e^{-f} \nabla f)] = Err_2 \quad (33)$$

with error term,

$$Err_2 = -2\hat{\chi} \cdot (\nabla^2 f - \nabla f \nabla f) - \nabla tr \chi \cdot \nabla f + (tr \chi \eta - 2\hat{\chi} \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega)) \cdot \nabla f,$$

where $\nabla_3, \nabla, \operatorname{div}$ are defined relative of the old foliation (u, \underline{u}) .

The proof is based on a straightforward but lengthy computation, which we postpone to the appendix.

2.3. Estimates for $\text{tr}\chi'$. In this subsection, we apply Proposition 2 to compute $\text{tr}\chi'$ up to an error term depending on δ_0 .

Equation (30) implies that

$$f = f_0, \quad (34)$$

By the Commutation Lemma 1,

$$\nabla_3 \nabla f + \frac{1}{2} \text{tr}\chi = O_{f_0}(\delta_0),$$

where $O_{f_0}(\delta_0)$ denotes a term bounded by $C\delta_0$, with C depending only on the L^∞ norms of f_0 , ∇f_0 and $\nabla^2 f_0$. By (17),

$$\nabla_3(r\nabla f) = O_{f_0}(\delta_0).$$

Therefore,

$$r\nabla f = r_0\nabla f_0 + O_{f_0}(\delta_0), \quad (35)$$

A similar application of the Commutation Lemma 1 gives

$$r^2\nabla^2 f = r_0^2\nabla^2 f_0 + O_{f_0}(\delta_0). \quad (36)$$

Now, from equation (31),

$$\nabla_3 F + \frac{1}{2} \text{tr}\chi F = \nabla f + O_{f_0}(\delta_0)$$

we deduce,

$$\nabla_3(rF) = r_0\nabla f_0 + O_{f_0}(\delta_0)$$

and therefore, since $F_0 = e^{-f_0}|\nabla v_0| = 0$,

$$r|F| = ur_0|\nabla f_0| + O_{f_0}(\delta_0). \quad (37)$$

We next calculate ∇F . Using the Commutation Lemma 1, we deduce,

$$\nabla_3 \nabla F + \text{tr}\chi \nabla F = \nabla^2 f + O_{f_0}(\delta_0).$$

Thus,

$$r^2|\nabla F| = ur_0^2|\nabla^2 f_0| + O_{f_0}(\delta_0).$$

Since $G = e^{-f}F$ we also deduce,

$$r^2|\nabla G| = r_0^2 e^{-f_0}|\nabla^2 f_0| + O_{f_0}(\delta_0).$$

Next we calculate $\text{div } G$ from (32) which we write in the form,

$$\nabla_3(\text{div } G) + \text{tr}\chi \text{div } G = \text{div } (e^{-f}\nabla f) + O_{f_0}(\delta_0). \quad (38)$$

On the other hand, (33) implies

$$r^2 \text{div } (e^{-f}\nabla f) = r_0^2 \text{div } (e^{-f_0}\nabla f_0) + O_{f_0}(\delta_0).$$

Therefore, (38) implies

$$r^2 \operatorname{div} G = -ur_0^2 \Delta(e^{-f_0}) + O_{f_0}(\delta_0). \quad (39)$$

Finally, going back to (29),

$$\begin{aligned} \operatorname{tr} \chi' &= \operatorname{tr} \chi - 2e^f \operatorname{div} (G) + \frac{2}{r} |F|^2 + O_{f_0}(\delta_0) \\ &= \operatorname{tr} \chi + \frac{2ur_0^2}{r^2} \left(-\Delta f_0 + \left[1 + \frac{u}{r}\right] |\nabla f_0|^2 \right) + O_{f_0}(\delta_0). \end{aligned}$$

We summarize the result in the following proposition.

Proposition 3. *Assume that MA1-MA4 are verified in the space-time region $\mathcal{D}(u_*, \delta)$ and f, v defined according to (27). Then for all $0 \leq u \leq u_*$ on the incoming null hypersurface \underline{H}_δ , the expansion $\operatorname{tr} \chi'$ of the v foliation verifies,*

$$\operatorname{tr} \chi'(u, \underline{u} = \delta, \omega) = \operatorname{tr} \chi(u, \underline{u} = \delta, \omega) + \frac{2ur_0^2}{r^2} \left(-\Delta f_0 + \left[1 + \frac{u}{r}\right] |\nabla f_0|^2 \right) + O_{f_0}(\delta_0). \quad (40)$$

2.4. Main equation. We now combine the results of Propositions 1 and 3. According to Proposition 1 we have,

$$\operatorname{tr} \chi(u, \underline{u} = \delta, \omega) = \frac{2}{r(u, 0)} - \frac{1}{r^2(u, 0)} \int_0^\delta |\hat{\chi}_0|^2(\underline{u}', \omega) d\underline{u}' + O_{f_0}(\delta_0).$$

Thus, inserting in (40), we have

$$\operatorname{tr} \chi'(u, \underline{u} = \delta, \omega) \leq \frac{2}{r} + \frac{2ur_0}{r^2} \left(-\Delta f_0 + \left[1 + \frac{u}{r}\right] |\nabla f_0|^2 \right) - \frac{1}{r^2} M_0 - O_{f_0}(\delta_0).$$

where $r = r(u, 0)$ and

$$M_0 = \int_0^\delta |\hat{\chi}_0|^2(\underline{u}') d\underline{u}'.$$

To proceed, recall that we choose the function f_0 such that the sphere $\{(u, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ corresponds to $\{\underline{u} = \delta, v = 1\}$. We express the values of u on the sphere $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ as a function along $S_{0,0}$ via the coordinate identification provided by (18) and (19). We denote this function by $U = U(f_0)$. In fact, since $v = ue^{f_0}$, we deduce that $U = e^{-f_0}$ on $\{\underline{u} = \delta, v = 1\}$. Moreover, since according to (17),

$$\frac{dr}{du} = -1 + O(\delta_0 r),$$

we can write,

$$R = r_0 - e^{-f_0} + O(\delta_0).$$

To have $\text{tr}\chi'$ negative on $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\} = \{\underline{u} = \delta, v = 1\}$ we need,

$$\frac{2}{R} + \frac{2U}{R^2} \left(-\Delta f_0 + \left[1 + \frac{U}{R}\right] |\nabla f_0|^2 \right) < \frac{1}{R^2} M_0 - O_{f_0}(\delta_0). \quad (41)$$

We now re-express (41) with respect to $R = R(f_0)$. We have,

$$\begin{aligned} \nabla R &= \frac{dR}{df}(f_0) \nabla f_0 \\ \Delta R &= \frac{dR}{df}(f_0) \Delta f_0 + \frac{d^2 R}{d^2 f}(f_0) |\nabla f_0|^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{dR}{df}(f_0) &= \frac{dr}{du} \cdot \frac{dU}{df} = -\nabla_u r \cdot e^{-f_0} \\ \frac{d^2 R}{d^2 f}(f_0) &= \nabla_u^2 r \cdot e^{-f_0} + \nabla_u r e^{-f_0} \end{aligned}$$

In view of formula (8) and the equation,

$$\nabla_3 \text{tr}\underline{\chi} + \frac{1}{2}(\text{tr}\underline{\chi})^2 = -2\underline{\omega} \text{tr}\underline{\chi} - |\hat{\underline{\chi}}|^2$$

we deduce,

$$\begin{aligned} \nabla_u(r \nabla_u r) &= \frac{1}{8\pi} \nabla_u \int_{S_{u,\underline{u}}} \Omega \text{tr}\underline{\chi} = \frac{1}{8\pi} \int_{S_{u,\underline{u}}} \Omega (e_3(\Omega \text{tr}\underline{\chi}) + \Omega \text{tr}\underline{\chi} \text{tr}\underline{\chi}) \\ &= \frac{1}{16\pi} \int_{S_{u,\underline{u}}} \Omega^2 \text{tr}\underline{\chi}^2 - \frac{1}{8\pi} \int_{S_{u,\underline{u}}} \Omega^2 |\hat{\underline{\chi}}|^2 = 1 + rO(\delta_0) \end{aligned}$$

Hence,

$$r \nabla_u^2 r + (\nabla_u r)^2 = 1 + O(\delta_0)$$

from which we deduce,

$$r \nabla_u^2 r = O(\delta_0). \quad (42)$$

Hence,

$$\begin{aligned} |\nabla R|^2 &= |\nabla f_0|^2 e^{-2f_0} |\nabla_u r|^2 \\ &= |\nabla f_0|^2 e^{-2f_0} + O_{f_0}(\delta_0) \\ \Delta R &= -\Delta f_0 e^{-f_0} \nabla_u r + (\nabla_u^2 r \cdot e^{-f_0} + \nabla_u r e^{-f_0}) |\nabla f_0|^2 \\ &= e^{-f_0} (\Delta f_0 - |\nabla f_0|^2) + O_{f_0}(\delta_0). \end{aligned}$$

Thus,

$$\begin{aligned} |\nabla f_0|^2 &= e^{2f_0} |\nabla R|^2 + O_{f_0}(\delta_0) \\ \Delta f_0 &= e^{f_0} \Delta R + |\nabla f_0|^2 + O_{f_0}(\delta_0) \\ &= e^{f_0} \Delta R + e^{2f_0} |\nabla R|^2 + O_{f_0}(\delta_0). \end{aligned}$$

Therefore, we have

$$\begin{aligned} & -\Delta f_0 + \left(1 + \frac{U}{R}\right) |\nabla f_0|^2 + \frac{R}{U} \\ &= -e^{f_0} \Delta R - e^{2f_0} |\nabla R|^2 + (1 + e^{-f_0} R^{-1}) e^{2f_0} |\nabla R|^2 + R e^{f_0} \\ &= e^{f_0} \left[-\Delta R - R^{-1} |\nabla R|^2 + R \right] \\ &= \frac{1}{U} \left[-\Delta R - R^{-1} |\nabla R|^2 + R \right]. \end{aligned}$$

Hence, multiplying by (41) by $\frac{R^2}{2}$, we deduce that (41) is equivalent to

$$-\Delta R + R^{-1} |\nabla R|^2 + R < \frac{1}{2} M_0 - O_{f_0}(\delta_0).$$

Re-expressing f_0 in terms of R , this is equivalent to

$$-\Delta R + R^{-1} |\nabla R|^2 + R < \frac{1}{2} M_0 - O_R(\delta_0),$$

where $O_R(\delta_0)$ denotes a term bounded by $C\delta_0$ such that C depends only on the L^∞ norm of R , ∇R and $\nabla^2 R$.

This concludes the proof of Theorem 2.

3. SOLUTIONS TO THE DEFORMATION EQUATION ON A FIXED SPHERE (S, γ) .

To prove our main Theorem 1 it suffices¹¹ now to show that if M_0 verifies the the assumption (15) then an appropriate solution to the differential inequality on the standard sphere $S = S_{0,0}$,

$$-\Delta R + R^{-1} |\nabla R|^2 + R < 2^{-1} M_0, \tag{43}$$

can be found. In our argument below we can also assume that u_* is close to r_0 .

Let $R = e^{-\phi}$ and for simplicity of notation rewrite $M = 2^{-1} M_0$. Then the main deformation equation (43) reduces to

$$\Delta \phi + 1 < M e^\phi. \tag{44}$$

¹¹Assuming $\delta\delta_0^{-1}$, δ_0 sufficiently small, depending on M_0 .

We show below that (44) can be solved as long as $M \geq 0$ and $M \geq c > 0$ on some open ball of S . Our approach provides an explicit construction using the Green's function for the Laplacian on S . The main observation is that given any function $\tilde{\phi}$, there exists a sufficiently large constant C such that (44) is satisfied by $\phi = \tilde{\phi} + C$ on the set where M has a positive lower bound. It is therefore sufficient first to construct a function $\tilde{\phi}$ satisfying (44) only on the complement of the set where M has a positive lower bound. It turns out that an appropriately rescaled and cut-off version of the Green's function for the Laplacian satisfies this property. As we will show below, this approach provides the sharp bounds for the solution ϕ when M is concentrated on a small geodesic ball $B_p(\epsilon)$.

We prove the following proposition, which together with Theorem 2, implies our main theorem (Theorem 1):

Proposition 4. *Let $M_* = \min_{B_p(\epsilon)} M$. Then there exists a function ϕ_{ϵ, M_*} verifying the inequality (44) and such that*

$$\phi_{\epsilon, M_*} \leq \log\left(\frac{r_0^2}{M_* \epsilon^2}\right) + O(1) \quad (45)$$

$$|\nabla \phi_{\epsilon, M_*}| = O(r_0^{-2} \epsilon^{-1}), \quad |\nabla^2 \phi_{\epsilon, M_*}| = O(r_0^{-2} \epsilon^{-2}). \quad (46)$$

Remark 4. In fact (45) is sharp up to a constant for the function

$$\begin{cases} M = 0 & \text{on } S \setminus B_p(\epsilon) \\ M = M_* & \text{on } B_p(\epsilon) \end{cases}$$

as can be seen by the following argument:

On $S \setminus B_p(\epsilon)$, we must have

$$\int_{S \setminus B_p(\epsilon)} (\Delta \phi + 1) < 0.$$

Since $\int_S \Delta \phi = 0$,

$$\int_{B_p(\epsilon)} \Delta \phi \geq c \int_{S \setminus B_p(\epsilon)} 1 \geq c r_0^2.$$

Thus there exists a point q in $B_p(\epsilon)$ such that

$$\Delta \phi(q) \geq \frac{c r_0^2}{\epsilon^2}.$$

At the same time (44) implies that

$$M_* e^{\phi(q)} \geq \Delta \phi(q) \geq \frac{c r_0^2}{\epsilon^2}.$$

Thus

$$\phi(q) \geq \log\left(\frac{c r_0^2}{M_* \epsilon^2}\right).$$

3.1. Proof of the main theorem. Returning to our task of constructing a trapped surface, note that the upper bound (45) for ϕ corresponds to our desired lower bound for R . More precisely, (45) implies that for some $C > 0$,

$$\frac{1}{R} = e^\phi \leq \frac{Cr_0^2}{M_*\epsilon^2}.$$

In particular, for M satisfying the assumption of Theorem 1, i.e.,

$$\inf_{B_p(\epsilon)} M \geq M_* > 0,$$

the proposition implies the existence of a function R verifying (43) and a lower bound depending only on $\frac{M_*\epsilon^2}{r_0^2}$. Therefore, for fixed $\frac{M_*\epsilon^2}{r_0^2}$, we can choose $\delta\delta_0^{-1}$ and δ_0 small enough such that u_* is sufficiently close to r_0 and $r_0 - u_* + C\delta_0 < \min R$. This guarantees that the sphere $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ lies within $\mathcal{D}(u_*, \delta)$. Moreover, given a function R verifying (43), the term $C\delta_0$ in (20) can be made arbitrarily small by choosing δ_0 small. Thus, by Theorem 2, the sphere defined by $\{(\underline{u}, r, \omega) : \underline{u} = \delta, r = R(\omega)\}$ is a trapped surface. Since $R \geq (r_0 - u_*)$, the constructed trapped surface has area at least $\approx (r_0 - u_*)^2$.

This concludes the proof of the main theorem. It thus remains to prove Proposition 4.

3.2. Proof of Proposition 4. To this end, we use the following standard lemma concerning the Green's function of the standard sphere S . A proof, which we sketch below for completeness, can be found in Theorem 4.13 in [1].

Lemma 2. *Given a smooth riemannian metric γ on S , there exists a function w , smooth outside the point p , such that*

$$\Delta_d w + \frac{1}{2r_0^2} = 2\pi\delta_p \tag{47}$$

where Δ_d is the distributional Laplacian associated to the metric γ and δ_p is the Dirac measure at p . Moreover, if λ_p denotes the distance function from p ,

$$w = \chi \log \lambda_p + v \tag{48}$$

with v smooth in $S \setminus \{p\}$ and satisfying

$$|v| \leq C, \quad |\nabla v| = o(\lambda^{-1}), \quad |\nabla^2 v| = o(\lambda^{-2});$$

and χ a smooth cut-off function identically equals to 1 in a small neighborhood of p .

Proof. We construct w by first finding an approximate solution to (47) and then controlling the error. Notice that in order to get good estimates in L^∞ for v and its derivatives, we cannot directly bound v . Instead, we need to give a more precise approximate solution (see (51) below).

Let $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bump function supported in $[0, \frac{i_*}{2}]$ such that

$$\chi(x) = 1 \quad \text{for } 0 \leq x \leq \frac{i_*}{4},$$

where i_* is the radius of injectivity on S .

Let λ_p be the distance function from p and f_p be a function defined away from p by

$$f_p = \Delta(\chi \log \lambda_p)$$

and equal to 0 at p .

It is easy to see that for every C^2 function φ

$$2\pi\varphi(p) = \int \chi(x) \log \lambda_p(x) \Delta\varphi(x) dV(x) + \int \varphi(x) f_p(x) dV(x),$$

and in particular

$$\int_S f_p(x) dV(x) = 2\pi, \quad \text{for all } p.$$

Moreover, for every continuous function φ ,

$$\Delta_x \left(\int \chi(y) \log \lambda_x(y) \varphi(y) dV(y) \right) = 2\pi\varphi(x) - \int f_x(y) \varphi(y) dV(y). \quad (49)$$

Define g_p by

$$g_p(x) = \frac{1}{2\pi} \int f_p(y) f_x(y) dV(y).$$

One checks that g_p is C^1 and that

$$\int_S g_p(x) dV(x) = 2\pi, \quad \text{for all } p.$$

Finally, define h_p by solving

$$\Delta h_p(x) = \frac{1}{2\pi} \int g_p(y) f_x(y) dV(y) - \frac{1}{2r_0^2}. \quad (50)$$

Notice that this can be solved since $\frac{1}{2\pi} \int g_p(y) f_x(y) dV(y) = 2\pi$ and the right hand side of (50) integrates to 0. By standard elliptic estimates, $h_p(x) \in C^2$.

Let

$$v(x) = \frac{1}{2\pi} \int f_p(y) \chi(y) \log \lambda_x(y) dV(y) + \frac{1}{2\pi} \int g_p(y) \chi(y) \log \lambda_x(y) dV(y) + h_p(x). \quad (51)$$

One easily checks that v obeys

$$\Delta v = f_p(y) - \frac{1}{2r_0^2},$$

which implies, by (49),

$$\Delta_d(\chi \log \lambda_p + v) = 2\pi\delta_p - \frac{1}{2r_0^2}$$

Moreover, v satisfies all the estimates in the conclusion of the lemma. \square

Using this lemma, we now proceed to the proof of Proposition 4:

Proof of Proposition 4. Consider the cut-off function

$$\begin{cases} \chi_\epsilon = 0 & \text{on } B_p(\epsilon/2) \\ \chi_\epsilon = 1 & \text{on } S \setminus B_p(\epsilon) \end{cases}$$

and define $\tilde{w}_\epsilon = \chi_\epsilon w - (1 - \chi_\epsilon) \log \epsilon$. Note that \tilde{w}_ϵ verifies the following properties:

$$\begin{cases} \tilde{w}_\epsilon = 0, & \text{on } B_p(\epsilon/2) \\ \tilde{w}_\epsilon = \log \epsilon + O(1), & \text{on } B_p(\epsilon) \setminus B_p(\epsilon/2) \\ \tilde{w}_\epsilon = \log \lambda + O(1), & \text{on } S \setminus B_p(\epsilon) \\ \nabla \tilde{w}_\epsilon = O(\epsilon^{-1}), & \text{on } S \setminus B_p(\epsilon/2) \\ \nabla^2 \tilde{w}_\epsilon = O(\epsilon^{-2}), & \text{on } S \setminus B_p(\epsilon/2) \\ \Delta \tilde{w}_\epsilon + \frac{1}{2r_0^2} = 0, & \text{on } S \setminus B_p(\epsilon) \end{cases} \quad (52)$$

Consider now the function $\tilde{\phi}_\epsilon = 3r_0^2 \tilde{w}_\epsilon$ and observe that, on $S \setminus B_p(\epsilon)$, we must have,

$$\Delta \tilde{\phi}_\epsilon + 1 = -\frac{3}{2} + 1 < 0.$$

Thus, we have,

$$\begin{cases} \tilde{\phi}_\epsilon = 0, & \text{on } B_p(\epsilon/2) \\ \tilde{\phi}_\epsilon = 3r_0^2 \log \epsilon + O(1), & \text{on } B_p(\epsilon) \setminus B_p(\epsilon/2) \\ \tilde{\phi}_\epsilon = 3r_0^2 \log \lambda + O(1), & \text{on } S \setminus B_p(\epsilon) \\ \nabla \tilde{\phi}_\epsilon = O(r_0^2 \epsilon^{-1}), & \text{on } S \setminus B_p(\epsilon/2) \\ \nabla^2 \tilde{\phi}_\epsilon = O(r_0^2 \epsilon^{-2}), & \text{on } S \setminus B_p(\epsilon/2) \\ \Delta \tilde{\phi}_\epsilon + 1 < 0, & \text{on } S \setminus B_p(\epsilon) \end{cases} \quad (53)$$

Finally, we let

$$\phi_{\epsilon, M_*} = -\log s_{\epsilon, M_*} + \tilde{\phi}_\epsilon, \quad (54)$$

where

$$s_{\epsilon, M_*} = \frac{cM_*\epsilon^{2+3r_0^2}}{r_0^2}, \quad \text{for some small constant } c.$$

Then, on $S \setminus B_p(\epsilon)$,

$$\Delta\phi_{\epsilon, M_*} + 1 = \Delta\tilde{\phi}_\epsilon + 1 < 0.$$

On $B_p(\epsilon)$,

$$\Delta\phi_{\epsilon, M_*} + 1 = \Delta\tilde{\phi}_\epsilon + 1 < \frac{Cr_0^2}{\epsilon^2} \leq M_* e^{\phi_{\epsilon, M_*}}.$$

Therefore, (44) is verified on S .

Moreover, (53) and (54) imply that ϕ_{ϵ, M_*} obeys the bounds asserted in the proposition. □

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APPENDIX A. PROOF OF PROPOSITION 2

In this appendix, we derive the transformation formula for $\text{tr}\chi$ for the change of foliation defined in Section 2.2.

Lemma 3. *The new incoming expansion $\text{tr}\chi'$ verifies the transformation formula,*

$$\text{tr}\chi' = \text{tr}\chi - 2e^f \text{div}(e^{-f}F) - \text{tr}\underline{\chi}|F|^2 - 4\underline{\chi}_{bc}F^bF^c - 2(\eta + \zeta) \cdot F \quad (55)$$

where $F_a = e^{-f}\Omega\nabla_a v$ and $\text{tr}\chi, \zeta, \text{tr}\underline{\chi}, \underline{\chi}, \underline{\omega}$ are connection coefficients for the given double null foliation (u, \underline{u}) .

Proof. We have,

$$\chi'(e'_a, e'_b) := g(D_{a'}e'_4, e'_b) = g(D_a e'_4, e'_b) - F_a g(D_3 e'_4, e'_b)$$

Now, writing $e'_4 = e_4 - 2F + |F|^2 e_3$ with $F = F_c e_c$ and $e'_b = e_b - F_b e_3$,

$$\begin{aligned} g(D_a e'_4, e'_b) &= g(D_a(e_4 - 2F + |F|^2 e_3), e_b - F_b e_3) \\ &= \chi(e_a, e_b) - 2F_b \zeta_a - 2\nabla_a F_b + 2F_b g(D_a F, e_3) + |F|^2 g(D_a e_3, e_b - F_b e_3) \\ &= \chi_{ab} - 2\zeta_a F_b - 2\nabla_a F_b - 2F_b \underline{\chi}(F, e_a) + |F|^2 \underline{\chi}_{ab} \\ &= \chi_{ab} - 2\zeta_a F_b - 2\nabla_a F_b - 2F_b F_c \underline{\chi}_{ac} + |F|^2 \underline{\chi}_{ab} \end{aligned}$$

Also,

$$\begin{aligned} g(D_3 e'_4, e'_b) &= g(D_3(e_4 - 2F + |F|^2 e_3), e_b - F_b e_3) \\ &= g(D_3 e_4, e_b) - F_b g(D_3 e_4, e_3) - 2\nabla_3 F_b \\ &= 2\eta_b + 4F_b \underline{\omega} - 2\nabla_3 F_b \end{aligned}$$

Hence,

$$\begin{aligned} \chi'_{ab} &= \chi_{ab} - 2\zeta_b F_a - 2\nabla_a F_b - 2F_b F_c \underline{\chi}_{bc} + |F|^2 \underline{\chi}_{ab} - F_a (2\eta_b + 4F_b \underline{\omega} - 2\nabla_3 F_b) \\ &= \chi_{ab} - 2\nabla_a F_b + 2F_a \nabla_3 F_b - 2\zeta_b F_a - 2F_a \eta_b + |F|^2 \underline{\chi}_{ab} - 2F_b F_c \underline{\chi}_{ac} - 4\underline{\omega} F_a F_b \end{aligned}$$

By symmetry in a, b we deduce the formula,

$$\begin{aligned} \chi'_{ab} = \chi_{ab} &- (\nabla_a F_b + \nabla_b F_a) + \nabla_3 (F_a F_b) - (\zeta_b + \eta_b) F_a - (\zeta_a + \eta_a) F_b \\ &+ |F|^2 \underline{\chi}_{ab} - F_b F_c \underline{\chi}_{ac} - F_a F_c \underline{\chi}_{bc} - 4\underline{\omega} F_a F_b \end{aligned} \quad (56)$$

and, taking the trace,

$$\begin{aligned} \text{tr} \chi' &= \text{tr} \chi - 2\text{div} F + \nabla_3 |F|^2 - 2(\eta + \zeta) \cdot F + (|F|^2 \text{tr} \underline{\chi} - 2\underline{\chi}_{bc} F^b F^c) - 4\underline{\omega} |F|^2 \\ &= \text{tr} \chi - 2\text{div} F + \nabla_3 |F|^2 - 2(\eta + \zeta) \cdot F - 2\hat{\chi}_{bc} F^b F^c - 4\underline{\omega} |F|^2 \end{aligned}$$

We next calculate $\nabla_3 |F|^2$ using (27) and the commutation formula

$$[\nabla_3, \nabla]h = (\nabla \log \Omega) \nabla_3 h - \underline{\chi} \cdot \nabla h$$

or,

$$[\nabla_u, \nabla]h = -\Omega \underline{\chi} \cdot \nabla h$$

Since $\nabla_u f = 0$ and $F = \Omega^{-1} e^{-f} \nabla v$ we deduce,

$$\begin{aligned} \nabla_u F_a &= \nabla_u (\Omega e^{-f} \nabla v) = \Omega e^{-f} \nabla_u \nabla v + \nabla_u \Omega e^{-f} \nabla v \\ &= \Omega e^{-f} \nabla \nabla_u v - \Omega e^{-f} \Omega \underline{\chi} \cdot \nabla v + \nabla_u \Omega e^{-f} \nabla v \\ &= \Omega \nabla f - \Omega^2 e^{-f} \underline{\chi} \cdot \nabla v + \nabla_u \Omega e^{-f} \nabla v \\ &= \Omega \nabla f - \Omega \underline{\chi} \cdot F - \Omega^{-1} \nabla_u \Omega F \end{aligned}$$

or,

$$\begin{aligned} \nabla_3 F &= \nabla f - \underline{\chi} \cdot F - \Omega^{-1} \nabla_3 \Omega F \\ &= \nabla f - \underline{\chi} \cdot F + 2\underline{\omega} F \end{aligned}$$

i.e.,

$$\nabla_3 F + \frac{1}{2} \text{tr} \underline{\chi} F = \nabla f - \hat{\chi} \cdot F + 2\underline{\omega} F \quad (57)$$

from which we derive,

$$\nabla_3 |F|^2 = -\text{tr} \underline{\chi} |F|^2 + 2F \cdot \nabla f - 2\hat{\chi}_{bc} F^b F^c + 4\underline{\omega} |F|^2$$

Therefore,

$$\begin{aligned}
\text{tr}\chi' &= \text{tr}\chi - 2\text{div } F - 2(\eta + \zeta) \cdot F - 2\underline{\hat{\chi}}_{bc}F^bF^c - 4\underline{\omega}|F|^2 \\
&- \text{tr}\chi|F|^2 + 2F \cdot \nabla f - 2\underline{\hat{\chi}}_{bc}F^bF^c + 4\underline{\omega}|F|^2 \\
&= \text{tr}\chi - 2\text{div } F + 2F \cdot \nabla f - \text{tr}\chi|F|^2 - 4\underline{\hat{\chi}}_{bc}F^bF^c - 2(\eta + \zeta) \cdot F \\
&= \text{tr}\chi - 2e^f \text{div } (e^{-f}F) - \text{tr}\chi|F|^2 - 4\underline{\hat{\chi}}_{bc}F^bF^c - 2(\eta + \zeta) \cdot F
\end{aligned}$$

as desired. □

To understand how $\text{tr}\chi'$ differs from $\text{tr}\chi$ it only remains to derive a transport equation for $\text{div } G$ with $G = e^{-f}F$ and for $\text{div } (e^{-f}\nabla f)$.

In view of (57) and $e_3(f) = 0$ we have for $G := e^{-f}F$.

$$\nabla_3 G + \frac{1}{2}\text{tr}\chi G = e^{-f}\nabla f - \underline{\hat{\chi}} \cdot G + 2\underline{\omega}G \quad (58)$$

To derive a transport equation for $\text{div } G$ we make use of the following

Lemma 4. *Assume that the S -tangent vectorfield V verifies an equation of the form,*

$$\nabla_3 V + \frac{1}{2}\text{tr}\chi V = -\underline{\hat{\chi}} \cdot V + W$$

Then,

$$\begin{aligned}
\nabla_3(\text{div } V) + \frac{1}{2}\text{tr}\chi \text{div } V &= \text{div } W + W \cdot \nabla(\log \Omega) - 2\underline{\hat{\chi}} \cdot \nabla V - \nabla \text{tr}\chi \cdot V \\
&+ (\text{tr}\chi \zeta - 2\underline{\hat{\chi}} \zeta - 2\underline{\hat{\chi}} \cdot \nabla(\log \Omega)) \cdot V
\end{aligned}$$

Proof.

$$\nabla_3(\text{div } V) + \frac{1}{2}\text{tr}\chi \text{div } V = \text{div } (-\underline{\hat{\chi}} \cdot V + W) - \frac{1}{2}\nabla \text{tr}\chi \cdot V + [\nabla_3, \text{div}]V$$

We make use of the commutation formula, see Lemma 1,

$$[\nabla_3, \text{div}]V = -\frac{1}{2}\text{tr}\chi \text{div } V - \underline{\hat{\chi}} \cdot \nabla V + (\underline{\beta} - \eta \cdot \underline{\hat{\chi}}) \cdot V + \nabla(\log \Omega) \cdot \nabla_3 V$$

Therefore,

$$\begin{aligned}
\nabla_3(\operatorname{div} V) + \operatorname{tr}\underline{\chi}\operatorname{div} V &= \operatorname{div} (-\hat{\chi} \cdot V + W) - \hat{\chi} \cdot \nabla V + \left(\underline{\beta} - \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} - \eta \cdot \hat{\chi}\right) \cdot V \\
&+ \nabla(\log \Omega) \cdot \nabla_3 V \\
&= \operatorname{div} W - 2\hat{\chi} \cdot \nabla V + \left(-\operatorname{div} \hat{\chi} + \underline{\beta} - \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} - \eta \cdot \hat{\chi}\right) \cdot V \\
&+ \nabla(\log \Omega) \cdot \left(-\frac{1}{2}\operatorname{tr}\underline{\chi}V - \hat{\chi} \cdot V + W\right) \\
&= \operatorname{div} W + W \cdot \nabla(\log \Omega) - 2\hat{\chi} \cdot \nabla V \\
&+ \left(-\operatorname{div} \hat{\chi} - \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} + \underline{\beta} - \eta \cdot \hat{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi}\nabla(\log \Omega) - \hat{\chi} \cdot \nabla(\log \Omega)\right) \cdot V
\end{aligned}$$

Using the Codazzi equation, $\operatorname{div} \hat{\chi} = \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} + \underline{\beta} + \zeta \cdot (\hat{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi})$ as well as $\eta = \zeta + \nabla(\log \Omega)$ we derive,

$$\begin{aligned}
&-\operatorname{div} \hat{\chi} - \frac{1}{2}\nabla\operatorname{tr}\underline{\chi} + \underline{\beta} - \eta \cdot \hat{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi}\nabla(\log \Omega) - \hat{\chi} \cdot \nabla(\log \Omega) \\
&= -\nabla\operatorname{tr}\underline{\chi} - \zeta \cdot (\hat{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi} - \eta \cdot \hat{\chi} - \frac{1}{2}\operatorname{tr}\underline{\chi}\nabla(\log \Omega) - \hat{\chi} \cdot \nabla(\log \Omega)) \\
&= -\nabla\operatorname{tr}\underline{\chi} - \hat{\chi} \cdot (\zeta + \eta + \nabla(\log \Omega)) + \frac{1}{2}\operatorname{tr}\underline{\chi}(\zeta + \nabla(\log \Omega)) \\
&= -\nabla\operatorname{tr}\underline{\chi} - 2\hat{\chi} \cdot (\zeta + \nabla(\log \Omega)) + \operatorname{tr}\underline{\chi}\eta
\end{aligned}$$

Hence,

$$\begin{aligned}
\nabla_3(\operatorname{div} V) + \operatorname{tr}\underline{\chi}\operatorname{div} V &= \operatorname{div} W + W \cdot \nabla(\log \Omega) - 2\hat{\chi} \cdot \nabla V - \nabla\operatorname{tr}\underline{\chi} \cdot V \\
&+ (\operatorname{tr}\underline{\chi}\eta - 2\hat{\chi} \cdot \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega)) \cdot V
\end{aligned}$$

as desired. □

Applying the lemma to equation (58) we derive,

$$\begin{aligned}
\nabla_3(\operatorname{div} G) + \operatorname{tr}\underline{\chi}\operatorname{div} G &= \operatorname{div} W + W \cdot \nabla(\log \Omega) - 2\hat{\chi} \cdot \nabla G - \nabla\operatorname{tr}\underline{\chi} \cdot G \\
&+ (\operatorname{tr}\underline{\chi}\eta - 2\hat{\chi} \cdot \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega)) \cdot G
\end{aligned}$$

with $W = e^{-f}\nabla f + 2\underline{\omega}G$. Thus,

$$\operatorname{div} W + W \cdot \nabla(\log \Omega) = \operatorname{div} (e^{-f}\nabla f) + e^{-f}\nabla(\log \Omega) \cdot \nabla f + 2\operatorname{div} (\underline{\omega}G) + 2\nabla(\log \Omega)\underline{\omega}G$$

We deduce the following transport equation for $\operatorname{div} G$,

$$\nabla_3(\operatorname{div} G) + \operatorname{tr}\underline{\chi}\operatorname{div} G = \operatorname{div} (e^{-f}\nabla f) + 2\underline{\omega}\operatorname{div} G + \operatorname{Err}_1 \quad (59)$$

with error term,

$$\begin{aligned} \text{Err}_1 &= e^{-f} \nabla(\log \Omega) \cdot \nabla f - 2\hat{\chi} \cdot \nabla G - \nabla \text{tr} \underline{\chi} \cdot G \\ &+ (\text{tr} \underline{\chi} \eta - 2\hat{\chi} \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega) + 2\nabla \underline{\omega} + 2\underline{\omega} \nabla \log \Omega) \cdot G \end{aligned}$$

In the same manner we deduce a transport equation for the principal term $\text{div}(e^{-f} \nabla f)$ on the right hand side of (59). Indeed, since $\nabla_3 f = 0$ we derive,

$$\nabla_3(\nabla f) + \frac{1}{2} \text{tr} \underline{\chi} \nabla f = -\hat{\chi} \nabla f$$

Therefore, using Lemma 4,

$$\begin{aligned} \nabla_3 \text{div}(e^{-f} \nabla f) + \text{tr} \underline{\chi} \text{div}(e^{-f} \nabla f) &= -2\hat{\chi} \cdot \nabla(e^{-f} \nabla f) - \nabla \text{tr} \underline{\chi} \cdot e^{-f} \nabla f \\ &+ (\text{tr} \underline{\chi} \eta - 2\hat{\chi} \zeta - 2\hat{\chi} \cdot \nabla(\log \Omega)) \cdot e^{-f} \nabla f. \end{aligned}$$

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